

# NON-COMMUTATIVE INTEGRABLE SYSTEMS ON $b$ -SYMPLECTIC MANIFOLDS

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**ABSTRACT.** In this paper we study non-commutative integrable systems on  $b$ -Poisson manifolds. One important source of examples (and motivation) of such systems comes from considering non-commutative systems on manifolds with boundary having the right asymptotics on the boundary. In this paper we describe this and other examples and we prove an action-angle theorem for non-commutative integrable systems on a  $b$ -symplectic manifold in a neighbourhood of a Liouville torus inside the critical set of the Poisson structure associated to the  $b$ -symplectic structure.

## 1. INTRODUCTION

A non-commutative integrable system on a symplectic manifold with boundary yields a non-commutative system on a class of Poisson manifolds called  $b$ -Poisson manifolds, as long as the asymptotics of the system satisfy certain conditions near the boundary.  $b$ -Poisson manifolds constitute a class of Poisson manifolds which recently has been studied extensively (see for instance [GMP11], [GMP12], [GMPS13] and [GLPR14]) and integrable systems on such manifolds have been the object of study in [KMS15], [KM16] and [DKM15].

In [LMV11] an action-angle coordinate for Poisson manifolds is proved on a neighbourhood of a regular Liouville torus. This theorem cannot be applied to a neighborhood of a Liouville torus contained inside the critical set of the Poisson structure where the rank of the bivector field is no longer maximal. In this paper we extend the techniques in [LMV11] to consider a neighbourhood of a Liouville torus inside the critical set of a  $b$ -Poisson manifolds thus proving an action-angle theorem for non-commutative systems on  $b$ -Poisson manifolds.

The action-angle theorem for non-commutative integrable systems for symplectic manifolds was proved by Nehoroshev in [N72]. Our proof follows a combination of techniques from [LMV11] with techniques native to  $b$ -symplectic geometry. As in [LMV11] the key point of the proof is to find

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a torus action attached to a non-commutative integrable system and extend the Darboux-Carathéodory coordinates in a neighbourhood of the invariant subset. The upshot is the use of  $b$ -symplectic techniques and toric actions on these manifolds [GMPS13], [GMPS2] as we did in [KMS15] and [KM16] for commutative systems on  $b$ -manifolds. The proof is a combination of the theory of torus actions with a refinement of the commutative proof by considering Cas-basic forms and working with them as a subcomplex of the  $b$ -De Rham complex. The action-angle theorem for commutative integrable systems on  $b$ -symplectic manifolds yields semilocal models as twisted cotangent lifts (see [KM16]). It is also possible to visualize the action-angle theorem for non-commutative systems using twisted cotangent lifts.

The organization of this paper is as follows: In Section 2 we introduce the basic tools that will be needed in this paper. In Section 3 we provide a list of examples which includes non-commutative systems on symplectic manifolds with boundary and examples obtained from group actions including twisted  $b$ -cotangent lifts. We end this section exploring the Galilean group as a source of non-commutative examples in  $b$ -symplectic manifolds. In Section 4 we state and prove the action-angle coordinate theorem for  $b$ -symplectic manifolds.

## 2. PRELIMINARIES

**2.1. Integrable systems and action-angle coordinates on Poisson manifolds.** A Poisson manifold is a pair  $(M, \Pi)$  where  $\Pi$  is a bivector field such that the associated bracket on functions

$$\{f, g\} := \Pi(df, dg), \quad f, g : M \rightarrow \mathbb{R}$$

satisfies the Jacobi identity. The Hamiltonian vector field of a function  $f$  is defined as  $X_f := \Pi(df, \cdot)$ . This allows us to formulate equations of motion just as in the symplectic setting, i.e. given a Hamiltonian function  $H$  we consider the flow of the vector field  $X_H$ . The concept of integrable systems is well understood in the symplectic context. A similar definition is possible in the Poisson setting and the famous Arnold-Liouville-Mineur theorem on the semilocal structure of integrable systems has its analogue in the Poisson context. Both commutative and non-commutative integrable systems on Poisson manifolds were studied in [LMV11].

**Definition 1** (Non-commutative integrable system on a Poisson manifold). *Let  $(M, \Pi)$  be a Poisson manifold of (maximal) rank  $2r$ . An  $s$ -tuple of functions  $F = (f_1, \dots, f_s)$  on  $M$  is a **non-commutative (Liouville) integrable system** of rank  $r$  on  $(M, \Pi)$  if*

- (1)  $f_1, \dots, f_s$  are independent (i.e. their differentials are independent on a dense open subset of  $M$ );
- (2) The functions  $f_1, \dots, f_r$  are in involution with the functions  $f_1, \dots, f_s$ ;
- (3)  $r + s = \dim M$ ;

- (4) The Hamiltonian vector fields of the functions  $f_1, \dots, f_r$  are linearly independent at some point of  $M$ .

Viewed as a map,  $F : M \rightarrow \mathbb{R}^s$  is called the **momentum map** of  $(M, \Pi, F)$ .

When all the integrals commute, i.e.  $r = s$ , then we are dealing with the conventional case of a commutative integrable system.

**Example 2** (A generic example). Consider the manifold  $\mathbb{T}^r \times \mathbb{R}^s$  with coordinates

$$(\theta_1, \dots, \theta_r, p_1, \dots, p_r, z_1, \dots, z_{s-r})$$

equipped with the Poisson structure

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \pi'$$

where  $\pi'$  is any Poisson structure on  $\mathbb{R}^{s-r}$ . Then the functions

$$(p_1, \dots, p_r, z_1, \dots, z_s)$$

define a non-commutative integrable system of rank  $r$ .

As we will see in Theorem 3 below, any non-commutative integrable system semilocally takes this form, more precisely in the neighborhood of a regular compact connected level set of its integrals  $(f_1, \dots, f_s)$ .

**2.1.1. Standard Liouville tori.** Let  $(M, \Pi, F)$  be a non-commutative integrable system of rank  $r$ . We denote the non-empty subset of  $M$  where the differentials  $df_1, \dots, df_s$  (resp. the Hamiltonian vector fields  $X_{f_1}, \dots, X_{f_r}$ ) are independent by  $\mathcal{U}_F$  (resp.  $M_{F,r}$ ).

On the non-empty open subset  $M_{F,r} \cap \mathcal{U}_F$  of  $M$ , the Hamiltonian vector fields  $X_{f_1}, \dots, X_{f_r}$  define an integrable distribution of rank  $r$  and hence a foliation  $\mathcal{F}$  with  $r$ -dimensional leaves, see [LMV11].

We will only deal with the case where  $\mathcal{F}_m$  is compact. Under this assumption,  $\mathcal{F}_m$  is a compact  $r$ -dimensional manifold, equipped with  $r$  independent commuting vector fields, hence it is diffeomorphic to an  $r$ -dimensional torus  $\mathbb{T}^r$ . The set  $\mathcal{F}_m$  is called a *standard Liouville torus* of  $F$ .

The action-angle coordinate theorem proved in [LMV11] (Theorem 1.1) gives a semilocal description of the Poisson structure around a standard Liouville torus of a non-commutative integrable system:

**Theorem 3 (Action-angle coordinate theorem for non-commutative integrable systems on Poisson manifolds).** Let  $(M, \Pi, F)$  be a non-commutative integrable system of rank  $r$ , where  $F = (f_1, \dots, f_s)$  and suppose that  $\mathcal{F}_m$  is a standard Liouville torus, where  $m \in M_{F,r} \cap \mathcal{U}_F$ . Then there exist  $\mathbb{R}$ -valued smooth functions  $(p_1, \dots, p_r, z_1, \dots, z_{s-r})$  and  $\mathbb{R}/\mathbb{Z}$ -valued smooth functions  $(\theta_1, \dots, \theta_r)$ , defined in a neighborhood  $U$  of  $\mathcal{F}_m$ , and functions  $\phi_{kl} = -\phi_{lk}$ , which are independent of  $\theta_1, \dots, \theta_r, p_1, \dots, p_r$ , such that

- (1) The functions  $(\theta_1, \dots, \theta_r, p_1, \dots, p_r, z_1, \dots, z_{s-r})$  define a diffeomorphism  $U \simeq \mathbb{T}^r \times B^s$ ;

- (2) The Poisson structure can be written in terms of these coordinates as,

$$\Pi = \sum_{i=1}^r \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{k,l=1}^{s-r} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};$$

- (3) The leaves of the surjective submersion  $F = (f_1, \dots, f_s)$  are given by the projection onto the second component  $\mathbb{T}^r \times B^s$ , in particular, the functions  $f_1, \dots, f_s$  depend on  $p_1, \dots, p_r, z_1, \dots, z_{s-r}$  only.

The functions  $\theta_1, \dots, \theta_r$  are called angle coordinates, the functions  $p_1, \dots, p_r$  are called action coordinates and the remaining coordinates  $z_1, \dots, z_{s-r}$  are called transverse coordinates.

**2.2.  $b$ -Poisson and  $b$ -symplectic manifolds.** A symplectic form  $\omega$  induces a Poisson structure  $\Pi$  defined via

$$\Pi(df, dg) = \omega(X_f, X_g)$$

where  $X_f, X_g$  are the Hamiltonian vector fields defined with respect to  $\omega$ . On the other hand, a Poisson structure which does not have full rank everywhere, i.e. the set of Hamiltonian vector fields spans the tangent space at every point, does not induce a symplectic structure. However, if the Poisson structure drops rank in a controlled way as defined below, it is possible to associate a so-called  $b$ -symplectic structure.

**Definition 4** ( $b$ -Poisson structure). *Let  $(M^{2n}, \Pi)$  be an oriented Poisson manifold. If the map*

$$p \in M \mapsto (\Pi(p))^n \in \bigwedge^{2n}(TM)$$

*is transverse to the zero section, then  $\Pi$  is called a  **$b$ -Poisson structure** on  $M$ . The hypersurface  $Z = \{p \in M \mid (\Pi(p))^n = 0\}$  is the **critical hypersurface** of  $\Pi$ . The pair  $(M, \Pi)$  is called a  **$b$ -Poisson manifold**.*

It is possible and convenient to work in the “dual” language of forms instead of bivector fields. The object equivalent to a  $b$ -Poisson structure will be a  $b$ -symplectic structure. To define  $b$ -symplectic structures and, in general,  $b$ -forms we introduce the concept of  $b$ -manifolds and the  $b$ -tangent bundle associated to the critical set  $Z$ :

**Definition 5.** *A  **$b$ -manifold** is a pair  $(M, Z)$  of an oriented manifold  $M$  and an oriented hypersurface  $Z \subset M$ . A  **$b$ -vector field** on a  $b$ -manifold  $(M, Z)$  is a vector field which is tangent to  $Z$  at every point  $p \in Z$ .*

The set of  $b$ -vector fields is a Lie subalgebra of the algebra of all vector fields on  $M$ . Moreover, if  $x$  is a local defining function for  $Z$  on some open set  $U \subset M$  and  $(x, y_1, \dots, y_{N-1})$  is a chart on  $U$ , then the set of  $b$ -vector fields on  $U$  is a free  $C^\infty(M)$ -module with basis  $(x \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_N})$ . A locally  $C^\infty(M)$ -module has a vector bundle associated to it. We call the vector bundle associated to the sheaf of  $b$ -vector fields the  **$b$ -tangent**

**bundle** denoted  ${}^bT^*M$ . The  $b$ -cotangent bundle  ${}^bT^*M$  is, by definition, the vector bundle dual to  ${}^bTM$ .

Given a defining function  $f$  for  $Z$ , let  $\mu \in \Omega^1(M \setminus Z)$  be the one-form  $\frac{df}{f}$ . If  $v$  is a  $b$ -vector field then the pairing  $\mu(v) \in C^\infty(M \setminus Z)$  extends smoothly over  $Z$  and hence  $\mu$  itself extends smoothly over  $Z$  as a section of  ${}^bT^*M$ . We will write  $\mu = \frac{df}{f}$ , keeping in mind that on  $Z$  the expression only makes sense when evaluated on  $b$ -tangent vectors.

**Definition 6** ( $b$ -de Rham- $k$ -forms). *The sections of the vector bundle  $\Lambda^k({}^bT^*M)$  are called  $b$ - $k$ -forms ( $b$ -de Rham- $k$ -forms) and the sheaf of these forms is denoted  ${}^b\Omega^k(M)$ .*

For  $f$  a defining function of  $Z$  every  $b$ - $k$ -form can be written as

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \quad (1)$$

The decomposition (1) enables us to extend the exterior  $d$  operator to  ${}^b\Omega^k(M)$  by setting

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$

The right hand side is well defined and agrees with the usual exterior  $d$  operator on  $M \setminus Z$  and also extends smoothly over  $M$  as a section of  $\Lambda^{k+1}({}^bT^*M)$ . Since we have  $d^2 = 0$ , we can define the differential complex of  $b$ -forms, the  $b$ -de Rham complex.

**Definition 7.** *Let  $(M^{2n}, Z)$  be a  $b$ -manifold and  $\omega \in {}^b\Omega^2(M)$  a closed  $b$ -form. We say that  $\omega$  is  $b$ -symplectic if  $\omega_p$  is of maximal rank as an element of  $\Lambda^2({}^bT_p^*M)$  for all  $p \in M$ .*

It was shown in [GMP12] that  $b$ -symplectic and  $b$ -Poisson manifolds are in one-to-one correspondence.

The classical Darboux theorem for symplectic manifolds has its analogue in the  $b$ -symplectic case:

**Theorem 8** ( $b$ -Darboux theorem [GMP12]). *Let  $(M, Z, \omega)$  be a  $b$ -symplectic manifold. Let  $p \in Z$  be a point and  $z$  a local defining function for  $Z$ . Then, on a neighborhood of  $p$  there exist coordinates  $(x_1, y_1, \dots, x_{n-1}, y_{n-1}, z, t)$  such that*

$$\omega = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z} dz \wedge dt.$$

The cohomology of the  $b$ -de Rham complex, whose groups are denoted by  ${}^bH^*(M)$ , can be understood from the classic de Rham cohomologies of  $M$  and  $Z$  via the Mazzeo-Melrose theorem:

**Theorem 9** (Mazzeo-Melrose). *The  $b$ -cohomology groups of  $M^{2n}$  satisfy*

$${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

Under the Mazzeo-Melrose isomorphism, a  $b$ -form of degree  $p$  has two parts: its first summand, the *smooth* part, is determined (by Poincaré duality) by integrating the form along any  $p$ -dimensional cycle transverse to  $Z$  (such an integral is improper due to the singularity along  $Z$ , but the principal value of this integral is well-defined). The second summand, the *singular* part, is the residue of the form along  $Z$ .

**2.3.  $b$ -functions.** It is convenient to enlarge the set of smooth functions to the set of  $b$ -functions  ${}^bC^\infty(M)$ , so that the  $b$ -form  $\frac{df}{f}$  is exact, where  $f$  is a defining function for  $Z$ . We define a  $b$ -function to be a function on  $M$  with values in  $\mathbb{R} \cup \{\infty\}$  of the form

$$c \log|f| + g,$$

where  $c \in \mathbb{R}$  and  $g$  is a smooth function. For ease of notation, from now on we identify  $\mathbb{R}$  with the completion  $\mathbb{R} \cup \{\infty\}$ .

We define the differential operator  $d$  on this space in the obvious way:

$$d(c \log|f| + g) := \frac{c df}{f} + dg \in {}^b\Omega^1(M),$$

where  $dg$  is the standard de Rham derivative.

As in the smooth case, we define the ( $b$ -)Hamiltonian vector field of a  $b$ -function  $f \in {}^bC^\infty(M)$  as the (smooth) vector field  $X_f$  satisfying

$$\iota_{X_f}\omega = -df.$$

Obviously, the flow of a  $b$ -Hamiltonian vector field preserves the  $b$ -symplectic form and hence the Poisson structure, so  $b$ -Hamiltonian vector fields are in particular Poisson vector fields.

**2.4. Twisted  $b$ -cotangent lift.** Given a Lie group action on a smooth manifold  $M$ ,

$$\rho : G \times M \rightarrow M : (g, m) \mapsto \rho_g(m)$$

we define the cotangent lift of the action to  $T^*M$  via the pullback:

$$\hat{\rho} : G \times {}^bT^*M \rightarrow {}^bT^*M : (g, p) \mapsto \rho_{g^{-1}}^*(p).$$

It is well-known that the lifted action  $\hat{\rho}$  is Hamiltonian with respect to the canonical symplectic structure on  $T^*M$  (see [GS90]).

We want to view the lifted action as a  $b$ -Hamiltonian action by means of a construction first described in [KM16].

Consider  $T^*S^1$  with standard coordinates  $(\theta, a)$ . We endow it with the following one-form defined for  $a \neq 0$ , which we call the logarithmic Liouville one-form in analogy to the construction in the symplectic case:  $\lambda_{tw,c} = \log|a|d\theta$  for  $a \neq 0$ .

Now for any  $(n-1)$ -dimensional manifold  $N$ , let  $\lambda_N$  be the classical Liouville one-form on  $T^*N$ . We endow the product  $T^*(S^1 \times N) \cong T^*S^1 \times T^*N$  with the product structure  $\lambda := (\lambda_{tw,c}, \lambda_N)$  (defined for  $a \neq 0$ ). Its negative differential  $\omega = -d\lambda$  extends to a  $b$ -symplectic structure on the whole manifold and the critical hypersurface is given by  $a = 0$ .

Let  $K$  be a Lie group acting on  $N$  and consider the component-wise action of  $G := S^1 \times K$  on  $M := S^1 \times N$  where  $S^1$  acts on itself by rotations. We lift this action to  $T^*M$  as described above. This construction, where  $T^*M$  is endowed with the  $b$ -symplectic form  $\omega$ , is called the **twisted  $b$ -cotangent lift**.

If  $(x_1, \dots, x_{n-1})$  is a chart on  $N$  and  $(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$  the corresponding chart on  $T^*N$  we have the following local expression for  $\lambda$

$$\lambda = \log |a| d\theta + \sum_{i=1}^{n-1} y_i dx_i.$$

Just as in the symplectic case, this action is Hamiltonian with moment map given by contracting the fundamental vector fields with the Liouville one-form  $\lambda$ .

### 3. NON-COMMUTATIVE $b$ -INTEGRABLE SYSTEMS

In [KMS15] we introduced a definition of integrable systems for  $b$ -symplectic manifolds, where we allow the integrals to be  $b$ -functions. Such a “ $b$ -integrable system” on a  $2n$ -dimensional manifold consists of  $n$  integrals, just as in the symplectic case. Here we introduce the definition for the more general non-commutative case:

**Definition 10** (Non-commutative  $b$ -integrable system). *A non-commutative  $b$ -integrable system of rank  $r$  on a  $2n$ -dimensional  $b$ -symplectic manifold  $(M^{2n}, \omega)$  is an  $s$ -tuple of functions  $F = (f_1, \dots, f_r, f_{r+1}, \dots, f_s)$  where  $f_1, \dots, f_r$  are  $b$ -functions and  $f_{r+1}, \dots, f_s$  are smooth such that the following conditions are satisfied:*

- (1) *The differentials  $df_1, \dots, df_s$  are linearly independent as  $b$ -cotangent vectors on a dense open subset of  $M$  and on a dense open subset of  $Z$ ;*
- (2) *The functions  $f_1, \dots, f_r$  are in involution with the functions  $f_1, \dots, f_s$ ;*
- (3)  *$r + s = 2n$ ;*
- (4) *The Hamiltonian vector fields of the functions  $f_1, \dots, f_r$  are linearly independent as smooth vector fields at some point of  $Z$ .*

We call the first  $r$  functions  $(f_1, \dots, f_r)$  the commuting part of the system and the last  $s - r$  functions the non-commuting part.

The case  $r = s = n$  where we are dealing with a commutative system was studied in [KMS15].

We denote the non-empty subsets of  $M$  where condition (1) resp. (4) are satisfied by  $\mathcal{U}_F$  resp.  $M_{F,r}$ . The points of the intersection  $M_{F,r} \cap \mathcal{U}_F$  are called *regular*. As in the general Poisson case, the Hamiltonian vector  $X_{f_1}, \dots, X_{f_r}$  fields define an integrable distribution of rank  $r$  on this set and we denote the corresponding foliation by  $\mathcal{F}$ . If the leaf through a point  $m \in M$  is compact, then it is an  $r$ -torus (“**Liouville torus**”), denoted  $\mathcal{F}_m$ .

*Remark 11.* In the symplectic case, if the differentials  $df_i (i = 1, \dots, r)$  are linearly independent at a point  $p$ , then also the corresponding Hamiltonian vector fields  $X_{f_i}$  are independent at  $p$ . However, the situation is more delicate in the  $b$ -symplectic case. The differentials  $df_i$  are  $b$ -one-forms. At a point  $p$  where the  $df_i$  are independent as  $b$ -cotangent vectors, the corresponding Hamiltonian vector fields  $X_{f_i}$  are independent at  $p$  as  $b$ -tangent vectors. However, for  $p \in Z$  the natural map  ${}^bTM|_p \rightarrow TZ|_p$  is not injective and therefore we cannot guarantee independence of the  $X_{f_i}$  as smooth vector fields. This is why the condition (4) is needed. As an example, consider  $\mathbb{R}^2$  with standard coordinates  $(t, z)$  and  $b$ -symplectic structure

$$\frac{1}{t} dt \wedge dz.$$

Then the function  $z$  has a differential  $dz$  which is non-zero at all points of  $\mathbb{R}^2$ , but the Hamiltonian vector field of  $z$  is  $t \frac{\partial}{\partial t}$  and vanishes along  $Z = \{t = 0\}$ . We do not allow this kind of systems in our definition, since we are interested precisely in the dynamics on  $Z$  and the existence of  $r$ -dimensional Liouville tori there. We remark that the definition has already been given in an analogous way for general Poisson manifolds in [LMV11].

#### 4. EXAMPLES OF (NON-COMMUTATIVE) $b$ -INTEGRABLE SYSTEMS

**4.1. Non-commutative integrable systems on manifolds with boundary.** In [KMS15] we introduced new examples of integrable systems using existing examples on manifolds with boundary. We can reproduce a similar scheme in the non-commutative case. As a concrete example, let the manifold with boundary be  $M = N \times H_+$ , where  $(N, \omega_N)$  is any symplectic manifold and  $H_+$  is the upper hemisphere including the equator. We endow the interior of  $H_+$  with the symplectic form  $\frac{1}{h} dh \wedge d\theta$ , where  $(h, \theta)$  are the standard height and angle coordinates and the interior of  $M$  with the corresponding product structure. Now let  $(f_1, \dots, f_s)$  be a non-commutative integrable system of rank  $r$  on  $N$ . Then on the interior of  $M$  we can, for instance, define the following (smooth) non-commutative integrable system:

$$(\log |h|, f_1, \dots, f_s)$$

Taking the double of  $M$  we obtain a non-commutative  $b$ -integrable system on  $N \times S^2$ .

**4.2. Examples coming from  $b$ -Hamiltonian  $\mathbb{T}^r$ -actions.** In [Bo03] it is shown how to construct integrable systems from the Hamiltonian action of a Lie group  $G$  on a *symplectic* manifold  $M$ : Let  $\mu : M \rightarrow \mathfrak{g}^*$  be the moment map of the action and consider the algebra of functions on  $M$  generated by  $\mu$ -basic functions and  $G$ -invariant functions. Then under certain assumptions, this algebra is *complete* in the sense of [Bo03], Definition 1.1 therein. This result is the content of Theorem 2.1 in [Bo03]. In our terminology, this means that the algebra of functions admits a basis of functions  $f_1, \dots, f_s$  which form a non-commutative integrable system on  $M$ . The assumptions



needed for this to hold are satisfied in particular when the action is proper, which is the case for any compact Lie group  $G$ .

This result can be used in the  $b$ -symplectic case to semilocally construct a non-commutative  $b$ -integrable system on a  $b$ -symplectic manifolds  $M^{2n}$  with an effective Hamiltonian  $\mathbb{T}^r$ -action as follows: Let us denote the critical hypersurface of  $M$  by  $Z$  and assume  $Z$  is connected. Let  $t$  be a defining function for  $Z$ . A Hamiltonian  $\mathbb{T}^r$ -action on a  $b$ -symplectic manifold, by definition, satisfies that the  $b$ -one-form  $\iota_X \# \omega$  is exact for all  $X \in \mathfrak{t}$ . We consider an action with the property that, moreover, for some  $X \in \mathfrak{t}$  the  $b$ -one-form  $\iota_X \# \omega$  is a genuine  $b$ -one-form, i.e. not smooth. Then the following proposition proved in [GMPS13] about the “splitting” of the action holds: The critical hypersurface  $Z$  is a product  $\mathcal{L} \times \mathbb{S}^1$ , where  $\mathcal{L}$  is a symplectic leaf inside  $Z$  and in a neighborhood of  $Z$  there is a splitting of the Lie algebra  $\mathfrak{t} \simeq \mathfrak{t}_Z \times \langle X \rangle$ , which induces a splitting  $\mathbb{T}^r \simeq \mathbb{T}_Z^{r-1} \times \mathbb{S}^1$  such that the  $\mathbb{T}_Z^{r-1}$ -action on  $Z$  induces a Hamiltonian  $\mathbb{T}_Z^{r-1}$ -action on  $\mathcal{L}$ . Let  $\mu_{\mathcal{L}} : \mathcal{L} \rightarrow \mathfrak{t}_Z^*$  be the moment map of the latter. Then on a neighborhood  $\mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \simeq \mathcal{U} \subset M$  of  $Z$  the  $\mathbb{T}^r$ -action has moment map

$$\begin{aligned} \mu_{\mathcal{U} \setminus Z} : \mathcal{L} \times \mathbb{S}^1 \times ((-\varepsilon, \varepsilon) \setminus \{0\}) &\rightarrow \mathfrak{t}^* \simeq \mathfrak{t}_Z^* \times \mathbb{R} \\ (\ell, \rho, t) &\mapsto (\mu_{\mathcal{L}}(\ell), c \log |t|). \end{aligned}$$

Let  $(f_1, \dots, f_s)$  be the non-commutative integrable system induced on  $\mathcal{L}$  by applying the theorem in [Bo03] to the  $\mathbb{T}^{r-1}$ -action on  $\mathcal{L}$ . This system has rank  $r-1$ . On a neighborhood  $\mathcal{L} \times \{-\delta < \theta < \delta\} \times \{-\epsilon < t < \epsilon\}$  it extends to a non-commutative  $b$ -integrable system  $(\log |t|, f_1, \dots, f_s)$  of rank  $r$ . The Liouville tori of the system are the orbits of the action.

**4.3. The geodesic flow.** A special case of a  $\mathbb{T}^r$ -action is obtained in the case of a Riemannian manifold  $M$  which is assumed to have the property that all its geodesics are closed. These manifolds are called P-manifolds. In this case the geodesics admit a common period (see e.g. [Be12], Lemma 7.11); hence their flow induces an  $S^1$ -action on  $M$ . In the same way the standard cotangent lift induces a system on  $T^*M$  we can use the twisted  $b$ -cotangent lift (see subsection 2.4) to obtain a  $b$ -Hamiltonian  $S^1$ -action on  $T^*M$  and hence a non-commutative  $b$ -integrable system on  $T^*M$ . In dimension two, examples of P-manifolds are Zoll and Tannery surfaces (see Chapter 4 in [Be12]).

**4.4. The Galilean group.** The Galilean group has its physical origin in the (non-relativistic) transformations between two reference frames which differ by relative motion at a constant velocity  $b$ . Together with spatial rotations and translations in time and space, this is the so-called (*inhomogeneous*) Galilean group  $G$ . We now present in detail this example as a non-commutative integrable system, see also [MM16].

We consider the evolution space

$$V = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \ni (t, x, y),$$

where  $t \in \mathbb{R}$  is time and  $x, y \in \mathbb{R}^3$  are the position and velocity respectively.

The Galilean group can be viewed as a Lie subgroup of  $\text{GL}(\mathbb{R}, 5)$  consisting of matrices of the form

$$\begin{pmatrix} A & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}, \quad A \in \text{SO}(3), b \in \mathbb{R}^3, c \in \mathbb{R}^3, e \in \mathbb{R}. \quad (2)$$

If we denote the matrix above by  $a$  then the action  $a_V$  of the Galilean group on  $V$  is defined as follows:

$$a_V(t, x, v) = (t^*, x^*, y^*)$$

where  $t^* = t + e$ ,  $x^* = Ax + bt + c$ ,  $y^* = Ay + b$ .

The Lie algebra  $\mathfrak{g}$  of  $G$  is given by the set of matrices [S70]:

$$\begin{pmatrix} j(\omega) & \beta & \gamma \\ 0 & 0 & \epsilon \\ 0 & 0 & 0 \end{pmatrix}, \quad \epsilon \in \mathbb{R}, \omega \in \mathbb{R}^3, \beta \in \mathbb{R}^3, \gamma \in \mathbb{R}^3.$$

Here,  $j$  is the map that identifies  $\mathbb{R}^3$  with  $\mathfrak{so}(3)$ . Now instead of letting  $G$  act on the evolution space  $\mathbb{R}^7$ , we consider the action on the “space of motions”  $\mathbb{R}^3 \times \mathbb{R}^3$ , which is obtained by fixing time,  $t = t_0$ . This space is symplectic with the canonical symplectic form and the action of  $G$  on it is Hamiltonian.

In the literature the following integrals of the action are considered [S70]: Consider the basis of  $\mathfrak{g}$  given by the union of the standard basis on each of its components  $\mathfrak{so}(3)$ ,  $\mathbb{R}^3$  (corresponding to spatial translation  $\gamma$ ),  $\mathbb{R}$  (corresponding to time translation  $\epsilon$ ) and the Galilei boost Lie algebra  $\mathbb{R}^3$  (corresponding to the shift in velocity  $\beta$ ). The corresponding integrals are, respectively, the components of the angular momentum  $J = x \times y$ , velocity vector  $y$  and position vector  $x$  and the energy  $E$ . This system is non-commutative.

We want to investigate the action of certain subgroups of  $G$  and construct  $b$ -versions of the integrable systems. We will consider the space of motions  $\mathbb{R}^6$  with coordinates  $(x, y)$  as described above and time  $t = 0$ .

**Subgroup given by  $A = \text{Id}$ .** First, consider the subgroup of matrices of the form (2) where  $A$  is the identity matrix  $\text{Id} \in \text{SO}(3)$ . Then we have an action of  $\mathbb{R}^6$  on itself; in coordinates  $(x, y)$  as above the action consists of shifts in the  $x$  and  $y$  directions. This action is Hamiltonian with moment map and given by the full set of coordinates  $(x_1, x_2, x_3, y_1, y_2, y_3)$ . Clearly, this defines a non-commutative integrable system (of rank zero).

**Subgroup  $\text{SO}(3) \times \mathbb{R}^3$ .** Now let  $c, e$  be constant; for the sake of simplicity we assume they are equal to zero. Consider the subgroup of  $G$  where only  $A \in \text{SO}(3)$  and  $b \in \mathbb{R}^3$  vary. Then the action on  $\mathbb{R}^6$  is given by

$$A \cdot (x, y) = (Ax, Ay + b). \quad (3)$$

First we want to see that the  $\text{SO}(3)$ -action is Hamiltonian. Consider the standard basis of the Lie algebra  $\mathfrak{so}(3)$  corresponding under  $j$  to the unit

vectors in  $\mathbb{R}^3$ :

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On  $\mathbb{R}^3$  they describe rotations around the  $x_1$ ,  $x_2$ - and  $x_3$ -axis respectively. The corresponding fundamental vector fields on  $\mathbb{R}^6$  are

$$\begin{aligned} e_1^\# &= x_3 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial y_3} - x_2 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial y_2}, \\ e_2^\# &= x_1 \frac{\partial}{\partial x_3} - y_3 \frac{\partial}{\partial y_1} - x_3 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_3}, \\ e_3^\# &= x_2 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial y_2} - x_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_1}. \end{aligned}$$

One checks that these vector fields are Hamiltonian with respect to the following functions:

$$f_1 = x_2 y_3 - x_3 y_2, \quad f_2 = x_3 y_1 - x_1 y_3, \quad f_3 = x_1 y_2 - x_2 y_1.$$

Note that the  $f_i$  are the components of angular momentum  $J = x \times y$ . Hence we have seen that the  $\text{SO}(3)$ -action is Hamiltonian. The commutators are:

$$\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2}) = x_1 y_2 - x_2 y_1 = f_3,$$

and similarly  $\{f_2, f_3\} = f_1$  and  $\{f_3, f_1\} = f_2$ .

Since the  $f_i$  do not commute we need additional functions to define an integrable system on  $\mathbb{R}^6$ . This is where the  $\mathbb{R}^3$  action, given by the parameter  $b$  in Equation (3) comes into play. It has fundamental vector fields  $\frac{\partial}{\partial y_i}$  and the corresponding Hamiltonian functions are the coordinates  $x_i$ . Together with the integrals  $f_i$  they form a non-commutative integrable system  $(f_1, f_2, f_3, x_1, x_2, x_3)$  of rank zero.

**Subgroup  $\mathbb{S}^1 \times \mathbb{R}^3 \times \mathbb{R}^3$ .** Above we have studied the  $\text{SO}(3)$  action on  $\mathbb{R}^6$ . Now we restrict to the  $\mathbb{S}^1$ -subgroup of  $\text{SO}(3)$  given by rotations around the  $x_1$ - and  $y_1$ -axis. The associated integral is  $f_1 = x_2 y_3 - x_3 y_2$ . To obtain a non-commutative integrable system of non-zero rank, we can e.g. add the functions  $x_2, x_3, y_2$ , which do not commute with  $f_1$ , and the function  $y_1$ , which commutes with all the other functions. Hence we have obtained a non-commutative integrable system  $(y_1, f_1, x_2, x_3, y_2)$  of rank one.

**Some  $b$ -versions of these constructions.** We view  $\mathbb{R}^6$  as a  $b$ -symplectic manifold with critical hypersurface given by  $Z = \{y_1 = 0\}$  and canonical  $b$ -symplectic structure

$$\frac{dy_1}{y_1} \wedge dx_1 + \sum_{i=2}^r dy_i \wedge dx_i.$$

We want to see if the actions of the subgroups above can be seen as Hamiltonian actions on the  $b$ -symplectic manifold  $\mathbb{R}^6$  (i.e. their fundamental vector fields are Hamiltonian with respect to the  $b$ -symplectic structure). We treat the above cases one by one:

- The system  $(x_1, x_2, x_3, y_1, y_2, y_3)$  translates into the non-commutative  $b$ -integrable system  $(x_1, x_2, x_3, \log |y_1|, y_2, y_3)$ , i.e. the Hamiltonian vector fields with respect to the  $b$ -symplectic structure are the same and the system fulfils the required independence and commutativity properties.
- The  $\mathrm{SO}(3) \times \mathbb{R}^3$  action with moment map  $(f_1, f_2, f_3, x_1, x_2, x_3)$  is *not* Hamiltonian with respect to the  $b$ -symplectic structure. Indeed, away from  $Z$ , the fundamental vector field of the  $\mathrm{SO}(3)$ -action above associated to the Lie algebra element  $e_2$  has Hamiltonian function

$$x_3 \log |y_1| - x_1 y_3,$$

but this does not extend to a  $b$ -function on  $\mathbb{R}^6$ .

- The system  $(y_1, f_1, x_2, x_3, y_2)$  translates into the non-commutative  $b$ -integrable system  $(\log |y_1|, f_1, x_2, x_3, y_2)$ ; the induced action is the same as in the smooth case. On the other hand, the smooth system where we replace  $y_1$  by  $x_1$ , i.e.  $(x_1, f_1, x_2, x_3, y_2)$ , does not have such an analogue in the  $b$ -setting. Indeed, with respect to the  $b$ -symplectic form, the Hamiltonian vector field of the first function  $x_1$  is  $y_1$  and vanishes on  $Z$ , so the Hamiltonian vector fields of these functions are nowhere independent on  $Z$ .

## 5. ACTION-ANGLE COORDINATES FOR NON-COMMUTATIVE $b$ -INTEGRABLE SYSTEMS

In Theorem 8 we recalled the action-angle coordinate theorem for non-commutative integrable systems on Poisson manifolds, which was proved in [LMV11]. For  $b$ -symplectic manifolds and the commutative  $b$ -integrable systems defined there, we have proved an action-angle coordinate theorem [KMS15], which is similar to the symplectic case in the sense that even on the hypersurface  $Z$  where the Poisson structure drops rank there is a foliation by Liouville tori (with dimension equal to the rank of the system) and a semi-local neighborhood with “action-angle coordinates” around them. The main goal of this paper is to establish a similar result in the non-commutative case, proving the existence of  $r$ -dimensional invariant tori on  $Z$  and action-angle coordinates around them.

**5.1. Cas-basic functions.** Consider a non-commutative  $b$ -integrable system  $F$  on any Poisson manifold  $(M, \Pi)$ , where we denote the Poisson bracket by  $\{\cdot, \cdot\}$ . Let  $V := F(M) \cap \mathbb{R}^s$  be the “finite” target space of the integrals  $F$ . If we want to emphasize the functions  $F$  we are referring to, we will also write  $V_F$ . The space  $V$  inherits a Poisson structure  $\{\cdot, \cdot\}_V$  satisfying the following property:

$$\{g, h\}_V \circ F = \{g \circ F, h \circ F\},$$

where  $g, h$  are functions on  $V$ . Note that the values of the brackets  $\{f_i, f_j\}$  on  $M$  uniquely define the Poisson bracket  $\{\cdot, \cdot\}_V$ .

An  $F$ -basic function on  $M$  is a function of the form  $g \circ F$ . The Poisson structure  $\{\cdot, \cdot\}_V$  allows us to define the following important class of functions:

**Definition 12** (Cas-basic function). *An  $F$ -basic function  $g \circ F$  is called **Cas-basic** if  $g$  is a Casimir function with respect to  $\{\cdot, \cdot\}_V$ , i.e. the Hamiltonian vector field of  $g$  on  $V$  is zero.*

We recall the following characterisation of Cas-basic functions proved in [LMV11] in the setting of integrable systems on Poisson manifolds. The proof in the  $b$ -case is the same.

**Proposition 13.** *A function is Cas-basic if and only if it commutes with all  $F$ -basic functions.*

## 5.2. Normal forms for non-commutative $b$ -integrable systems.

**Definition 14** (Equivalence of non-commutative  $b$ -integrable systems). *Two non-commutative  $b$ -integrable systems  $F$  and  $F'$  are equivalent if there exists a Poisson map*

$$\mu : V_F \rightarrow V_{F'}$$

*taking one to the other:  $F' = \mu \circ F$ . Here,  $\mu$  is a Poisson map with respect to the Poisson structures induced on  $V_F$  and  $V_{F'}$  as defined in the previous section.*

We will not distinguish between equivalent systems: if the action-angle coordinate theorem that we will prove holds for one system then it holds for all equivalent systems too.

We prove a first “normal form” result for non-commutative  $b$ -integrable systems:

**Proposition 15.** *Let  $(M, \omega)$  be a  $b$ -symplectic manifold of dimension  $2n$  with critical hypersurface  $Z$ . Given a non-commutative  $b$ -integrable system  $F = (f_1, \dots, f_s)$  of rank  $r$  there exists an equivalent non-commutative  $b$ -integrable system of the form  $(\log |t|, f_2, \dots, f_s)$  where  $t$  is a defining function of  $Z$  and the functions  $f_2, \dots, f_s$  are smooth.*

*Proof.* First, assume that one of the functions  $f_1, \dots, f_r$  is a genuine  $b$ -function, without loss of generality  $f_1 = g + c \log |t'|$  where  $c \neq 0$  and  $t'$  a defining function of  $Z$ . Dividing  $f_1$  by the constant  $c$  and replacing the defining function  $t'$  by  $t := e^{gt'}$ , we can restrict to the case  $f_1 = \log |t|$ . We subtract an appropriate multiple of  $f_1$  from the other functions  $f_2, \dots, f_r$  so that they become smooth. Note that this does not affect their independence nor the commutativity condition for  $f_1, \dots, f_r$ , since  $f_1$  commutes with all the integrals. Also, since these operations do not affect the non-commutative part of the system, the induced Poisson bracket on the target space (cf. Section 5.1) remains unchanged. Hence we have obtained an equivalent  $b$ -integrable system of the desired form.

If all the functions  $f_1, \dots, f_s$  are smooth then from the independence of  $df_i$  ( $i = 1, \dots, s$ ) as  $b$ -one-forms on the set of regular points  $\mathcal{U}_F \cap M_{F,r}$  it

follows that

$$df_1 \wedge \dots \wedge df_s \wedge dt \neq 0 \in \Omega_p^s \quad \text{for } p \in \mathcal{U}_F \cap M_{F,r}, \quad (4)$$

where  $t$  is a defining function of  $Z$ . Therefore the functions  $f_1, \dots, f_s, t$  define a submersion on  $\mathcal{U}_F \cap M_{F,r}$  whose level sets are  $(r-1)$ -dimensional. On the other hand, the Hamiltonian vector fields  $X_{f_1}, \dots, X_{f_r}$  are linearly independent (on  $\mathcal{U}_F \cap M_{F,r}$ ) and tangent to the leaves of this submersion, because  $f_1, \dots, f_r$  commute with all  $f_j, j = 1, \dots, s$  and also with  $t$ , since any Hamiltonian vector field is tangent to  $Z$ . Contradiction.  $\square$

*Remark 16.* Recall that the Liouville tori of a non-commutative  $b$ -integrable system  $F$  are, by definition, the leaves of the foliation induced by  $X_{f_i}, i = 1, \dots, r$  on  $\mathcal{U}_F \cap M_{F,r}$ . A Liouville torus that intersects  $Z$  lies inside  $Z$ , since the Hamiltonian vector fields are Poisson vector fields and therefore tangent to  $Z$ . Moreover, since at least one of the first  $r$  integrals  $f_1, \dots, f_r$  has non-vanishing “log” part, the Liouville tori inside  $Z$  are *transverse* to the symplectic leaves.

We now prove a normal form result which holds semilocally around a Liouville torus. It describes the topology of the system: we will see that semilocally the foliation of Liouville tori is a product  $\mathbb{T}^r \times B^s$ , but the result does not yet give information about the Poisson structure.

**Proposition 17.** *Let  $m \in Z$  be a regular point of a non-commutative  $b$ -integrable system  $(M, \omega, F)$ . Assume that the integral manifold  $\mathcal{F}_m$  through  $m$  is compact (i.e. a torus  $\mathbb{T}^r$ ). Then there exist a neighborhood  $U \subset \mathcal{U}_F \cap M_{F,r}$  of  $\mathcal{F}_m$  and a diffeomorphism*

$$\phi : U \simeq \mathbb{T}^r \times B^s,$$

*which takes the foliation  $\mathcal{F}$  induced by the system to the trivial foliation  $\{\mathbb{T}^r \times \{b\}\}_{b \in B^n}$ .*

*Proof.* As described in the previous proposition, we can assume that our system has the form  $(\log |t|, f_2, \dots, f_s)$  where  $f_2, \dots, f_s$  are smooth. Consider the submersion

$$\tilde{F} := (t, f_2, \dots, f_s) : \mathcal{U}_F \rightarrow \mathbb{R}^s$$

which has  $r$ -dimensional level sets. The Hamiltonian vector fields  $X_{f_1}, \dots, X_{f_r}$  are tangent to the level sets. By comparing dimensions we see that the level sets of  $\tilde{F}$  are precisely the Liouville tori spanned by  $X_{f_1}, \dots, X_{f_r}$ .

Now, as described in [LMV11](Prop. 3.2) for classical non-commutative integrable systems, choosing an arbitrary Riemannian metric on  $M$  defines a canonical projection  $\psi : U \rightarrow \mathcal{F}_m$ . Setting  $\phi := \psi \times \tilde{F}$  we have a commuting

diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\phi} & \mathbb{T}^r \times B^s \\
 & \searrow \tilde{F} & \downarrow \pi \\
 & & B^s
 \end{array} \tag{5}$$

where

$$\pi = (\pi_1, \dots, \pi_s) : \mathbb{T}^r \times B^s \rightarrow B^s$$

is the canonical projection.

The change does not affect the Poisson structure on the target space. The commuting diagram (5) implies that

$$F = \underbrace{(\log |\pi_1|, \pi_2, \dots, \pi_s)}_{=: \pi'} \circ \phi$$

so the Poisson structure on the target space  $V = F(U) = \pi'(\mathbb{T}^r \times B^s)$  induced by  $F$  and  $\pi'$  is the same.  $\square$

The upshot is that for the semi-local study of non-commutative  $b$ -integrable systems around a Liouville torus we can restrict our attention to systems on  $(\mathbb{T}^r \times B^s, \omega)$  where  $\omega$  is the  $b$ -symplectic structure induced by the diffeomorphism  $\phi$  in the proof above and where the integrals  $F = (f_1, \dots, f_s)$  are given by

$$f_1 = \log |\pi_1|, f_2 = \pi_2, \dots, f_s = \pi_s,$$

where  $\pi_1, \dots, \pi_s$  are the projections on to the components of  $B^s$  and where we assume that the  $b$ -symplectic structure has exceptional hypersurface  $\{\pi_1 = 0\}$ . Also, we can assume that the system is regular on the whole manifold  $M = \mathbb{T}^r \times B^s$ . We refer to this system as the *standard non-commutative  $b$ -integrable system* on  $\mathbb{T}^r \times B^s$ .

*Remark 18.* The previous result gives a semilocal description of the manifold and the integrals. However, no information is given about the symplectic structure. In contrast, the action-angle coordinate theorem will specify the integrable system with respect to the canonical  $b$ -symplectic form ( $b$ -Darboux form) on  $\mathbb{T}^r \times B^s$ .

**5.3. Darboux-Carathéodory theorem.** The following is a key ingredient for the proof of the action-angle coordinate theorem. It tells us that we can locally extend a set of independent commuting functions to a  $b$ -Darboux chart.

**Lemma 19 (Darboux-Carathéodory theorem for  $b$ -integrable systems).** *Let  $m$  be a point lying inside the exceptional hypersurface  $Z$  of a  $b$ -symplectic manifold  $(M^{2n}, \omega)$ . Let  $t$  be a local defining function of  $Z$  around  $m$ . Let  $f_1, \dots, f_k$  be a set of commuting  $C^\infty$  functions with differentials that are linearly independent at  $m$  as elements of  ${}^bT_m^*(M)$ . Then there exist,*

on a neighborhood  $U$  of  $m$ , functions  $g_1, \dots, g_k, t, p_2, \dots, p_{n-k}, q_1, \dots, q_{n-k}$ , such that

- (a) The  $2n$  functions  $(f_1, g_1, \dots, f_k, g_k, t, q_1, p_1, q_2, \dots, p_{n-k}, q_{n-k})$  form a system of coordinates on  $U$  centered at  $m$ .
- (b) The  $b$ -symplectic form  $\omega$  is given on  $U$  by

$$\omega = \sum_{i=1}^k df_i \wedge dg_i + \frac{1}{t} dt \wedge dq_1 + \sum_{i=2}^{n-k} dp_i \wedge dq_i.$$

*Proof.* Let us denote the  $b$ -Poisson structure dual to  $\omega$  by  $\Pi$ . From the Darboux-Carathéodory Theorem for non-commutative integrable systems on Poisson manifolds it follows that on a neighborhood  $U$  of  $m$  we can complete the functions  $f_1, \dots, f_k$  to a coordinate system

$$(f_1, g_1, \dots, f_k, g_k, z_1, \dots, z_{2n-2r+2})$$

centred at  $m$  such that the  $b$ -Poisson structure reads

$$\Pi = \sum_{i=1}^k \frac{\partial}{\partial f_i} \wedge \frac{\partial}{\partial g_i} + \sum_{i,j=1}^{2n-2k} \phi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

for some functions  $\phi_{ij}$ . The image of the coordinate functions is an open subset of  $\mathbb{R}^{2n}$ ; we can assume that it is a product  $U_1 \times U_2$  where  $U_2$  corresponds to the image of  $z_1, \dots, z_{2n-2k}$ . Then

$$\Pi_2 = \sum_{i,j=1}^{2n-2r+2} \phi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

is a  $b$ -Poisson structure on  $U_2$  and hence by the  $b$ -Darboux theorem (Theorem 8), there exist coordinates on  $U_2$

$$(t, q_1, p_2, q_2, \dots, p_{n-k}, q_{n-k}),$$

where  $t$  is the local defining function for  $Z$  that we fixed in the beginning, such that

$$\Pi_2 = t \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial q_1} + \sum_{i=2}^{n-r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}.$$

The result follows immediately.  $\square$

*Remark 20.* A different proof can be given using the tools of [KMS15].

**5.4. Action-angle coordinates.** Let  $(M^{2n}, \omega, F)$  be a non-commutative  $b$ -integrable system of rank  $r$ . Let  $p \in M_{F,r} \cap \mathcal{U}_F$  be a regular point of the system lying inside the critical hypersurface and let  $\mathcal{F}_p$  be the Liouville torus passing through  $p$ . For a semilocal description of the system around  $\mathcal{F}_p$ , by Proposition 17 we can assume that we are dealing with the “standard model” of a non-commutative  $b$ -integrable system, i.e. the manifold is the cylinder  $\mathbb{T}^r \times B^s$  with some  $b$ -symplectic form  $\omega$  whose critical hypersurface is  $Z = \{\pi_1 = 0\} = \mathbb{T}^r \times \{0\} \times B^{s-1}$  and the integrals are  $f_1 = \log |\pi_1|$ ,  $f_i = \pi_i$ ,  $i = 2, \dots, r$ . Let  $c$  be the modular period of  $Z$ .



**Theorem 21.** *Then on a neighborhood  $W$  of  $\mathcal{F}_m$  there exist  $\mathbb{R} \setminus \mathbb{Z}$ -valued smooth functions*

$$\theta_1, \dots, \theta_r$$

*and  $\mathbb{R}$ -valued smooth functions*

$$t, a_2, \dots, a_r, p_1, \dots, p_\ell, q_1, \dots, q_\ell$$

*where  $\ell = n - r = \frac{s-r}{2}$  and  $t$  is a defining function of  $Z$ , such that*

- (1) *The functions  $(\theta_1, \dots, \theta_r, t, a_2, \dots, a_r, p_1, \dots, p_{n-r}, q_1, \dots, q_{n-r})$  define a diffeomorphism  $W \simeq \mathbb{T}^r \times B^s$ .*
- (2) *The  $b$ -symplectic structure can be written in terms of these coordinates as*

$$\omega = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^r d\theta_i \wedge da_i + \sum_{k=1}^{\ell} dp_k \wedge dq_k.$$

- (3) *The leaves of the surjective submersion  $F = (f_1, \dots, f_s)$  are given by the projection onto the second component  $\mathbb{T}^r \times B^s$ , in particular, the functions  $f_1, \dots, f_s$  depend on  $t, a_2, \dots, a_r, p_1, \dots, p_\ell, q_1, \dots, q_\ell$  only.*

*The functions*

$$\theta_1, \dots, \theta_r$$

*are called angle coordinates, the functions*

$$t, a_2, \dots, a_r$$

*are called action coordinates and the remaining coordinates*

$$p_1, \dots, p_{n-r}, q_1, \dots, q_{n-r}$$

*are called transverse coordinates.*

We will need the following two lemmas for the proof of this theorem:

**Lemma 22.** *Let  $F : M \rightarrow \overline{R}^s$  be an  $s$ -tuple of  $b$ -functions on the  $b$ -symplectic manifold  $M = \mathbb{T}^r \times B^s$ . If the coefficients of a vector field of the form  $Z = \sum_{j=1}^r \psi_j X_{f_j}$  are  $F$ -basic and the vector field has period one, then the coefficients are Cas-basic.*

*Proof.* The proof is exactly the same as in [LMV11] replacing Hamiltonian by  $b$ -Hamiltonian vector field.  $\square$

The following lemma was proved in [LMV11] (see Claim 2),

**Lemma 23.** *If  $\mathcal{Y}$  is a complete vector field of period one and  $P$  is a bivector field for which  $L_{\mathcal{Y}}^2 P = 0$ , then  $L_{\mathcal{Y}} P = 0$ .*

We can now proceed with the proof of Theorem 21:

*Proof.* (of Theorem 21) In the first step we perform “uniformization of periods” similar to [LMV11] and [KMS15]. The joint flow of the vector fields  $X_{f_1}, \dots, X_{f_r}$  defines an  $\mathbb{R}^r$ -action on  $M$ , but in general not a  $\mathbb{T}^r$ -action, although it is periodic on each of its orbits  $\mathbb{T}^r \times \{\text{const}\}$ .

Denoting the time- $s$  flow of the Hamiltonian vector field  $X_f$  by  $\Phi_{X_f}^s$ , the joint flow of the Hamiltonian vector fields  $X_{f_1}, \dots, X_{f_r}$  is

$$\begin{aligned} \Phi : \mathbb{R}^r \times (\mathbb{T}^r \times B^s) &\rightarrow \mathbb{T}^r \times B^s \\ ((s_1, \dots, s_r), (x, b)) &\mapsto \Phi_{X_{f_1}}^{s_1} \circ \dots \circ \Phi_{X_{f_r}}^{s_r}(x, b). \end{aligned}$$

Because the  $X_{f_i}$  are complete and commute with one another, this defines an  $\mathbb{R}^r$ -action on  $\mathbb{T}^r \times B^s$ . When restricted to a single orbit  $\mathbb{T}^r \times \{b\}$  for some  $b \in B^s$ , the kernel of this action is a discrete subgroup of  $\mathbb{R}^r$ , hence a lattice  $\Lambda_b$ , called the *period lattice* of the orbit  $\mathbb{T}^r \times \{b\}$ . Since the orbit is compact, the rank of  $\Lambda_b$  is  $r$ . We can find smooth functions (after shrinking the ball  $B^s$  if necessary)

$$\lambda_i : B^s \rightarrow \mathbb{R}^r, \quad i = 1, \dots, r$$

such that

- $(\lambda_1(b), \lambda_2(b), \dots, \lambda_r(b))$  is a basis for the period lattice  $\Lambda_b$  for all  $b \in B^s$
- $\lambda_i^1$  vanishes along  $\{0\} \times B^{s-1}$  for  $i > 1$ , and  $\lambda_1^1$  equals the modular period  $c$  along  $\{0\} \times B^{s-1}$ . Here,  $\lambda_i^j$  denotes the  $j^{\text{th}}$  component of  $\lambda_i$ .

Using these functions  $\lambda_i$  we define the “uniformized” flow

$$\begin{aligned} \tilde{\Phi} : \mathbb{R}^r \times (\mathbb{T}^r \times B^s) &\rightarrow (\mathbb{T}^r \times B^s) \\ ((s_1, \dots, s_r), (x, b)) &\mapsto \Phi\left(\sum_{i=1}^r s_i \lambda_i(b), (x, b)\right). \end{aligned}$$

The period lattice of this  $\mathbb{R}^r$ -action is constant now (namely  $\mathbb{Z}^r$ ) and hence the action naturally defines a  $\mathbb{T}^r$  action. In the following we will interpret the functions  $\lambda_i$  as functions on  $\mathbb{T}^r \times B^s$  (instead of  $B^s$ ) which are constant on the tori  $\mathbb{T}^r \times \{b\}$ .

We denote by  $Y_1, \dots, Y_r$  the fundamental vector fields of this action. Note that  $Y_i = \sum_{j=1}^r \lambda_i^j X_{f_j}$ . We now use the Cartan formula for  $b$ -symplectic forms (where the differential is the one of the complex of  $b$ -forms [GMP12]<sup>1</sup>) to compute the following expression:

$$\mathcal{L}_{Y_i} \mathcal{L}_{Y_i} \omega = \mathcal{L}_{Y_i} (d(\iota_{Y_i} \omega) + \iota_{Y_i} d\omega) \quad (6)$$

$$= \mathcal{L}_{Y_i} \left( d \left( - \sum_{j=1}^n \lambda_i^j df_j \right) \right) \quad (7)$$

$$= -\mathcal{L}_{Y_i} \left( \sum_{j=1}^n d\lambda_i^j \wedge df_j \right) = 0 \quad (8)$$

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<sup>1</sup>The decomposition of a  $b$ -form of degree  $k$  as  $\omega = \frac{dt}{t} \wedge \alpha + \beta$  for  $\alpha, \beta$  De Rham forms proved in [GMP12] allows to extend the Cartan formula valid for smooth De Rham forms to  $b$ -forms.

where in the last equality we used the fact that  $\lambda_i^j$  are constant on the level sets of  $F$ . By applying Lemma 23 this yields  $\mathcal{L}_{Y_i}\omega = 0$ , so the vector fields  $Y_i$  are Poisson vector fields, i.e. they preserve the  $b$ -symplectic form.

We now show that the  $Y_i$  are Hamiltonian, i.e. the  $(b)$ -one-forms

$$\alpha_i := \iota_{Y_i}\omega = - \sum_{j=1}^r \lambda_i^j df_j, \quad i = 1, \dots, r, \quad (9)$$

which are closed (because  $Y_i$  are Poisson) have a  $({}^b C^\infty)$ -primitive  $a_i$ . Since  $\lambda_i^1$  vanishes along  $\mathbb{T}^r \times \{0\} \times B^{s-1}$  for  $i > 1$ , the one-forms  $\alpha_i$  defined in Equation (9) and hence the functions  $a_i$  are smooth for  $i > 1$ . On the other hand,  $\lambda_1^1$  equals the modular period  $c$  along  $\mathbb{T}^r \times \{0\} \times B^{s-1}$  and therefore  $a_1 = c \log |t|$  for some defining function  $t$ .

We compute the functions  $a_2, \dots, a_r$  explicitly by applying a homotopy formula to the smooth one-forms  $\alpha_2, \dots, \alpha_r$ . This not only yields that these one-forms are exact but moreover that their  $C^\infty$ -primitives  $a_2, \dots, a_r$  are Cas-basic. (For the  $b$ -function  $a_1 = c \log |t|$  this is clear.) This is equivalent to proving that these closed forms are exact for the corresponding sub-complex of Cas-basic  $b$ -forms. We do this by means of adapted homotopy operators.

Consider the following homotopy formula (see for instance [MS12]):

$$\alpha_i - \phi_0^*(\alpha_i) = I(\underbrace{d(\alpha_i)}_{=0}) + d(I(\alpha_i)), \quad i = 2, \dots, r$$

where the functional  $I$  will be defined below and  $\phi_\tau$  is the retraction from  $\mathbb{T}^r \times B^s$  to  $\mathbb{T}^r \times \{0\} \times B^{s-r}$ :

$$\phi_\tau(x_1, \dots, x_r, b_1, \dots, b_r, b_{r+1}, \dots, b_s) = (x, \tau b_1, \dots, \tau b_r, b_{r+1}, \dots, b_s).$$

Note that  $\phi_0^*(\alpha_i) = 0$  since for any vector field  $X \in \mathcal{X}(\mathbb{T}^r \times \{0\} \times B^{s-r})$  we have  $\alpha_i(X) = 0$ . Recall that  $\alpha_i$  is a linear combination of  $d\pi_2, \dots, d\pi_r$  and therefore evaluates to zero for  $X$  a linear combination of  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial \pi_{r+1}}, \dots, \frac{\partial}{\partial \pi_s}$ . Therefore the homotopy formula tells us that the Hamiltonian function of  $\alpha_i$  ( $i = 2, \dots, r$ ) is explicitly given by  $I(\alpha_i)$ , which is defined as follows:

$$I(\alpha_i) = \int_0^1 \phi_\tau^*(\iota_{\xi_\tau}(\alpha_i)).$$

Here  $\xi_\tau$  is the vector field associated with the retraction:

$$\xi_\tau = \frac{d\phi_\tau}{d\tau} \circ \phi_\tau^{-1} = \frac{1}{\tau} \sum_{k=1}^s \pi_k \frac{\partial}{\partial \pi_k}.$$

Therefore we have

$$\iota_{\xi_\tau}(\alpha_i) = \frac{1}{\tau} \sum_{j=2}^r \lambda_i^j d\pi_j(\xi_\tau) = \frac{1}{\tau} \sum_{j=2}^r \sum_{k=1}^s \lambda_i^j \pi_k d\pi_j \left( \frac{\partial}{\partial \pi_k} \right) = \frac{1}{\tau} \sum_{j=2}^r \lambda_i^j \pi_j.$$

In the last equality we have used  $d\pi_j(\frac{\partial}{\partial \pi_k}) = \delta_{jk}$  for  $j > 2$ .

The projections  $\pi_j, j = 1, \dots, r$ , are obviously Cas-basic. The functions  $\lambda_i^j$  are Cas-basic by Lemma 22. The pullback  $\phi_\tau^*$  does not affect the Cas-basic property since it leaves the non-commutative part of the system invariant. We conclude that the functions  $\phi_\tau^*(\iota_{\xi_\tau}(\alpha_i))$  and hence  $a_1, \dots, a_r$  are Cas-basic.

We apply the Darboux-Carathéodory theorem for  $b$ -integrable systems to a point  $p \in \mathbb{T}^r \times \{0\}$  and the independent commuting smooth functions  $a_2, \dots, a_n$ . Then on a neighborhood  $U$  of  $p$  we obtain a set of coordinates  $(t, g_1, a_2, g_2, \dots, a_r, g_r, q_1, p_1, q_2, p_2, \dots, q_\ell, p_\ell)$ , where  $\ell = (s - 2r)/2$ , such that

$$\omega|_U = \frac{c}{t} dt \wedge dg_1 + \sum_{i=2}^k da_i \wedge dg_i + \sum_{i=1}^{\ell} dp_i \wedge dq_i. \quad (10)$$

The idea of the next steps is to extend this local expression to a neighborhood of the Liouville torus using the  $\mathbb{T}^r$ -action given by the vector fields  $X_{a_k}$ . First, note that the functions  $(q_1, p_1, q_2, p_2, \dots, q_\ell, p_\ell)$  do not depend on  $f_i$  and therefore can be extended to the saturated neighborhood  $W := \pi^{-1}(\pi(U))$ . Note that  $Y_i = \frac{\partial}{\partial g_i}$  and therefore the flow of the fundamental vector fields of the  $Y_i$ -action corresponds to translations in the  $g_i$ -coordinates. In particular, we can naturally extend the functions  $g_i$  to the whole set  $W$  as well.

We want to see that the functions

$$t, g_1, a_2, g_2, \dots, a_r, g_r, q_1, p_1, q_2, p_2, \dots, q_\ell, p_\ell \quad (11)$$

which are defined on  $W$ , indeed define a chart there (i.e. they are independent) and that  $\omega$  still has the form given in Equation (10).

It is clear that  $\{a_i, g_j\} = \delta_{ij}$  on  $W$ . To show that  $\{g_i, g_j\} = 0$ , we note that this relation holds on  $U$  and flowing with the vector fields  $X_{a_k}$  we see that it holds on the whole set  $W$ :

$$X_{a_k}(\{g_i, g_j\}) = \{\{g_i, g_j\}, a_k\} = \{g_i, \delta_{kj}\} - \{g_j, \delta_{ik}\} = 0.$$

This verifies that  $\omega$  has the form (10) above and in particular, we conclude that the derivatives of the functions (11) are independent on  $W$ , hence these functions define a coordinate system.

Since the vector fields  $\frac{\partial}{\partial g_i}$  have period one, we can view  $g_1, \dots, g_r$  as  $\mathbb{R}/\mathbb{Z}$ -valued functions (“angles”) and therefore use the letter  $\theta_i$  instead of  $g_i$ .  $\square$

*Remark 24.* In the language of cotangent models introduced in [KM16], this theorem can be expressed as saying that a non-commutative  $b$ -integrable system is semilocally equivalent given by the the twisted  $b$ -cotangent lift of the  $\mathbb{T}^r$ -action on itself by translations.

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